Fall 2014 MATH 24 Linear Algebra
Midterm Exam II

Instructions:

(1) Exam time: 169 hours.
(2) There are 10 problems in this exam and they are graded 2 points each.
(3) You can use books or existing internet resources, but you are not allowed to receive help from a human-being, either in person or via internet.
(4) All notations are standard.

Problem 1. *Find a matrix* $A \in M_{3\times 3}(\mathbb{R})$ *such that* $A^{2014} = 2014A$ *but* A^i *is not a scalar multiple of* A *for* $i = 2, 3, ..., 2013$.

Problem 2. *Let* A, B *be* $n \times n$ *matrices such that* $rank(A) + rank(B) < n$. *Show that* A *and* B *must have a common eigenvalue and a common eigenvector.*

Problem 3. *Let* T *be a linear transformation on a finite dimensional vector space* V. *Show that* T *is invertible if and only if it always maps a basis to a basis.*

Problem 4. *If an* $n \times n$ *matrix* X *commutes with every* $n \times n$ *matrix, i.e.,* $XY = YX$ *for every* $Y \in M_{n\times n}(F)$, *show that* X *must be* kI_n *for some scalar* $k \in F$.

Problem 5. *A matrix* A *is* symmetric *if* $A = A^t$ *and* skew-symmetric *if* $A = -A^t$. *Show that every matrix is a* **unique** *sum of the form* $B + C$ *where* B *is symmetric and* C *is skew-symmetric if and only if the scalar field does not have characteristic* 2.

Problem 6. *Let* V *be a* n*-dimensional vector space. Let* T *be a linear transformation from* V *to* V, *and let* v *be a vector in* V. *Show that if the whole space* V *is generated by* $\{v, T(v), T(T(v)), T(T(T(v))), \ldots\}$, *then it is generated by the first* n *elements of this list of vectors, i.e., by* $\{v, T(v), T(T(v)), T(T(T(v))), \ldots, T^{n-1}(v)\}$.

Problem 7. *Let* V *and* W *be finite-dimensional vector spaces. Let* T *and* U *be linear transformations from* V *to* W. *Show that there is a linear transformation* $S : V \to V$ *such that* $T \circ S = U$ *if and only if the range of* U *is a subspace of the range of* T.

Problem 8. *If* $A^2 = A$, *we call* A *an* idempotent *matrix. Show that every square matrix* M *can be written as a product of two matrices* $M = PA$ *where* P *is invertible and* A *is idempotent.*

Problem 9. *Let* $A, B \in M_{2\times 2}(\mathbb{R})$ *such that* $AB = BA$. *Prove or disprove:* A *and* B *have a common eigenvector.*

Problem 10. *Let* f *be a linear transformation from* $M_{2\times 2}(\mathbb{R})$ *to* $\mathbb{R}$ *such that for all* $A, B \in M_{2\times 2}(F)$, $f(AB) = f(BA)$, *and* $f(I_2) = 2$. *Prove or disprove:* f *must be the trace function.*